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All Facets of the Cut Cone $\mathbf{C}_n$ for $n = 7$ are Known

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The cut cone $\mathbf{C}_n$ is the cone generated by the characteristic vectors of all cuts of a complete graph K_n on n vertices. A detailed description of the cut cone $\mathbf{C}_n$ can be found in [1]. We use here notions and notations from [1]; in particular concerning roots, switching and permutation operation.

An important problem is to find facets of the cone $\mathbf{C}_n$. The cone $\mathbf{C}_1$ is empty, $\mathbf{C}_2$ is a ray and the cone $\mathbf{C}_3$ is a 3-dimensional simplicial cone, the 3 facets of which are defined by triangle inequalities.

All facets of the cones $\mathbf{C}_4$ and $\mathbf{C}_5$ were found by M. Deza [2], and all facets of $\mathbf{C}_6$ by D. Avis using computer check. Note that all facets of the cones $\mathbf{C}_n$ for $n \leqslant 6$ are hypermetric.

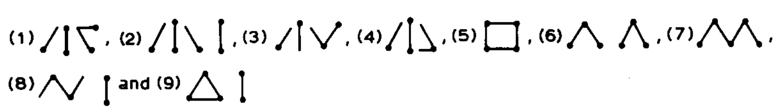
The cone $\mathbf{C}_7$ is the first cone for which there exist non-hypermetric facets. There are more than 40 000 facets for the cone $\mathbf{C}_7$. Therefore complete enumeration of all facets of $\mathbf{C}_7$ by computer is computationally infeasible. In fact, it is enough to compute all facets up to permutation and switching [1]. Also, moderate time and space resources are sufficient for computing all facets containing some fixed extreme rays (cuts) of the cone $\mathbf{C}_7$.

A list of 11 types of facets of the cone $\mathbf{C}_7$ (up to permutation and switching) is described in [1]. It is proved here that this list is complete.

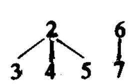
Recall that every ray $\delta(S)$ of the cut cone $\mathbf{C}_n$ is uniquely determined by a set S of cardinality $|S|$ such that $0 \leqslant |S| \leqslant [n/2]$. So, all cuts of $\mathbf{C}_7$ are determined by 7 singletons $\{l\}$, $1 \leqslant i \leqslant 7$, 21 pairs $\{ij\}$, $1 \leqslant i, j \leqslant 7$, and 35 triples, $\{ijk\}$, $1 \leqslant i, j, k \leqslant 7$, $i \neq f \neq k \neq l$. The cut lying in a facet is called root of the facet.

Below we describe 5 collections, (a), (b), (c), (d) and (e), of roots such that it is possible to compute all facets of $\mathbf{C}_7$ containing the roots of these collections. Then we prove that any facet of the cone $\mathbf{C}_7$ is switching or permutation equivalent to one of the computed facets.

(a) The first collection consists of sets of 4 roots of the form $\{1ij\}$. Actually, we choose a set of 4 triples, since computing all facets containing 3 triples needs too much time. There is one-to-one correspondence between sets of 4 triples of the form $\{1ij\}$ and graphs with at most 6 vertices and 4 edges $\{ij\}$. There are 9 non-isomorphic types of such graphs:



For example, the graph



corresponds to following 4 triples; $\{123\}$, $\{124\}$, $\{125\}$ and $\{167\}$. Note that if we

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switch this sets of roots by the root $\{167\}$ (i.e. we take the symmetric difference with $\{167\}$, e.g. $\{123\} \triangle \{167\} = \{2367\} = \{1234567\} - \{145\}$), we obtain (up to complementation) the set of roots $\{145\}$, $\{134\}$, $\{135\}$, $\{167\}$ of type (9). This implies that all facets which contain the 4 roots of type (2) are, in fact, switching equivalent to all facets containing the 4 roots of type (9).

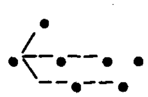
We obtained by computer that the numbers of facets containing each of the above types of 4 roots are as follows: (1) 77, (2) 81, (3) 219, (4) 188, (5) 589, (6) 118, (7) 206, (8) 62 and (9) 81.

(b) The second collection consists of the set of 3 roots $\{12\}$, $\{13\}$ and $\{23\}$ forming a triangle. There are 66 facets containing these roots. Note that corresponding cut vectors $\delta(12)$, $\delta(13)$, $\delta(23)$ satisfy the following identity:

$$\delta(12) + \delta(23) + \delta(13) = \delta(1) + \delta(2) + \delta(3) + \delta(123).$$

This equality implies that all facets containing the roots $\delta(12)$, $\delta(23)$ and $\delta(13)$ also contain the roots $\delta(1)$, $\delta(2)$, $\delta(3)$ and $\delta(123)$.

(c) The third collection is a set of 6 pairs forming the following tree:

T_0: 

By enumeration, we found 56 facets containing these 6 roots.

(d) The fourth collection is a set of 4 pairs covering the same vertex; for example, the pairs $\{12\}$, $\{13\}$, $\{14\}$ and $\{15\}$. We found 175 facets containing these 4 roots.

(e) The fifth and last collection is the set of 5 singletons, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$. If we switch a facet containing these 5 roots by the root $\{1\}$, we obtain a facet containing as roots $\{1\}$, $\{12\}$, $\{13\}$, $\{14\}$ and $\{15\}$. Therefore, all facets containing the above 5 singletons as roots belong, in fact, to the class of facets containing the collection (d) as the set of roots. It follows that there is no need to compute facets containing the collection (e) as the set of roots.

We obtained that all facets containing collections (a), (b), (c) or (d) as roots are, in fact, of one of the 11 types listed in [1]. We now prove that this list is complete, i.e. we show that every facet of $\mathbf{C}_7$ (up to switching and permutation) contains as roots one of sets of collections (a), (b), (c) or (d).

Each of the computed facets was of the type represented in the list of [1]. Now the completeness of that list can be proved as follows.

Suppose, for contradiction, that there is a facet which is not switching or permutation equivalent to any of the computed facets. Let R be the set of its roots.

Since R does not contain any set of roots from collection (a), all triples of R cover each vertex at most 3 times. What is a maximal cardinality of such a set of triples? Let A be the incidence matrix of vertices and triples. It has 7 rows corresponding to the vertices, and 35 columns corresponding to the triples. The value of the integer programming problem

$$\max\left\{\sum_{t=1}^{35} x_t \colon Ax \leqslant 3 \text{ and } x_t = 0, 1\right\}$$

is obviously the maximal number in question. The LP-relaxation of this problem gives an upper bound to this number. Since the LP-problem and its dual

$$\min\left\{\sum_{i=1}^{7} 3y_i : yA \geqslant 1,\ y \geqslant 0\right\}$$

have the solutions $x_t = \frac{1}{5}$, $1 \leqslant t \leqslant 35$, $y_i = \frac{1}{3}$, $1 \leqslant i \leqslant 7$, with the same value 7 of the functionals, 7 is its optimal value. It follows that the set R has at most 7 triples.

Since R does not contain collection (e) of roots, the set R has at most 4 singletons. Now, consider the graph generated by pairs which belong to R. Since R does not contain collections (b), (d) and (c) we deduce, respectively, that this graph has no triangle, its degree is at most 3 and it does not contain the tree T_0 as a spanning tree. It is not difficult to show that such a graph has at most 7 edges.

So, the set R must contain at most 7 triples, 7 pairs and 4 singletons; that is, at most $7 + 7 + 4 = 18$ roots, which is less than 20, the rank of a facet of the cone $\mathbf{C}_7$. Therefore all facets of the cone $\mathbf{C}_7$ have a switching or permutation equivalent facet among computed ones.

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